## GMP I <br> Operational Systems | Logic \| Number Theory | Probability | Statistics

## Operational Systems, Logic, \& Number Theory

Contents: As you work through the problems in this book, you will discover that operational systems, logic, number theory, probability, and statistics have been integrated into a mathematical whole. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records. Specifically, you should dedicate a section of your notebook (or a separate notebook) for keeping a record of the key vocabulary, including what the words mean as well as examples that help clarify the meaning for you. Key vocabulary is easy to identify since the words appear in italics within the problems.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will motivate your work in class. Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

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## Operational Systems, Logic, \& Number Theory

1. ROT13 fox. ROT13 is the name of a certain method for creating secret messages. The ROT13 method takes any English word or sentence and converts it into a secret language. Ask Wolfram|Alpha to translate a few messages using ROT13. Can you figure out how ROT13 encodes messages? Whether you can or you can't, invent your own way to encode messages in a way that would be easy for someone to figure out. Explain your coding procedure in detail and show how fox and pond are written in your secret language.
2. If you multiply any whole number by 2 , you get an even number. By trying enough examples, convince yourself that this is true. Then explain why it must be true.
3. The figure below, referred to as Key 5, shows one way to encode messages. Here's how it works. You take a letter like $s$ and you assign it the number that goes along with it on the spiral. In this case, $s$ becomes 18. You then multiply the number by 2 , obtaining a new number, in this case 36. You then go back to the spiral and see what letter goes with that new number. In this case, our 36 goes with $h$. So in this coding procedure, the letter $s$ becomes the letter $h$. We call this coding procedure the Key $5 \rightarrow$ Multiply by $2 \rightarrow$ Key 5 procedure.


Figure 1: Key 5
Use it to code the following words. Notice that some letters correspond to commas, periods, or spaces.
A. elephant
B. computer
C. facebook
4. How would you decode a message that was coded with the Key $5 \rightarrow$ Multiply by 2 $\rightarrow$ Key 5 procedure. Explain your method using words and complete sentences. Then show that it works for decoding the three words in the last problem. Finally, use your procedure to decode bfijviw.

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1. In the diagram below, some circles are closer to each other than other circles. Which two circles are the closest? Which two are the farthest apart? What is the average distance between two circles? Work in millimeters.

2. Looking at the Key 5 spiral, find two numbers that correspond to the letter ' N '. Find two numbers that correspond to the letter ' T '. Find two numbers that correspond to the letter 'A'. For each letter, subtract the smaller number from the larger. Explain any patterns you find.
3. (Continuation) Imagine extending the Key 5 spiral for as long as we needed. If we could do this, there would be an infinite list of numbers that could be associated with each letter. List the first five numbers that could be associated with the letter 'B'. Find a quick way to generate the same type of list for any letter.
4. Are there any letters or symbols that remain unchanged when using the Key $5 \rightarrow$ Multiply by $2 \rightarrow$ Key 5 procedure? Explain.
5. The numbers 0-28 make up the first 'winding' of the Key 5 spiral. The numbers 29-57 make up the second winding of the spiral. If we extended the spiral, which winding would each of the following numbers be on?
A. 68
B. 198
C. 1,921

Which letter or symbol would each number correspond to?
6. (Continuation) $68 / 29$. Solve the following division problems using Wolfram|Alpha. Instead of expressing your answer as a decimal, express your answer as a whole number plus a remainder.
A. $68 \div 29=$ ?
B. $198 \div 29=$ ?
C. $1,921 \div 29=$ ?

Explain the connection between this problem and the last one.
7. Which number on the 4 th winding corresponds to 13 on the first winding of the Key 5 spiral? Which number on the 10th winding corresponds to 23 on the first winding? Which number on the 100th winding corresponds to 2 on the first winding?

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1. What is the value of $3 \cdot 11$ ? Your answer corresponds to a certain number on the inner winding of Key 5. Which number? Because of these last two results, we say that $3 \cdot{ }_{29} 11=4$. Likewise, since $12 \cdot 9=108$ and 108 corresponds to 21 on the inner winding, we say that $12 \cdot{ }_{29} 9=21$. What is the value of $6 \cdot{ }_{29} 8$ ?
2. (Continuation) Just like the symbol • stands for multiplication, the symbol $\cdot 29$ stands for multiplication modulo 29 or multiplication mod 29 for short. And just as we read $6 \cdot 7$ as ' 6 times 7 ', we read $6 \cdot{ }_{29} 7$ as ' 6 times $7 \bmod$ 29'. Translate each of the following into words.
A. $6 \cdot{ }_{29} 7=13$
B. $5 \cdot{ }_{29} 4=20$
C. $8 \cdot{ }_{29} 8=9$

Two out of the three statements above are true and the other is false. Determine which is which.
3. The first number of the first winding is 0 . The first number of the second winding is 29 . What is the first number of the third, fourth, and fifth windings? Find a way to quickly calculate the first number of any winding.
4. Use the order of operations to calculate the value of each of the following.
A. $3 \cdot 29+9$
B. $2 \cdot 29+11$
C. $5 \cdot 29+27$

For each of your answers, determine what Key 5 winding number they are on and the number they correspond to on the first winding. Could these have been predicted before the order of operations were applied? Explain.
5. distance from chicago to miami. Consider the cities Chicago, Miami, Philadelphia, and Pittsburgh. What is the average distance between these cities? Do you think this average distance is close to the average distance you would get if you considered all of the cities in the United States? Explain.
6. Develop a quick way to find the answer to any mod 29 multiplication problem. Your method should work for all such problems. Illustrate your method by showing how to use it to find the value of $13 \cdot 2951$.
7. An agent for the Sikinian Secret Service told his chief that he intended to code his messages using a Key $5 \rightarrow$ Multiply Mod 29 by $\mathbf{3 0} \rightarrow$ Key 5 coding procedure. He was promptly fired. Why?
8. If someone were asked the question 'What is 4 times 2 ?', why might ' 8 ' be a better answer than ' $16 \div 2$ ', $2^{3}$, or $\sqrt{64}$ ?

## Operational Systems, Logic, \& Number Theory

1. The figure below is a clock face that has been modified by replacing the 12 with a zero. If the clock showed 6 o'clock, what would it show 3 hours later?; 6 hours later?; 11 hours later?


Figure 2: Modified Clock 12 Face
2. (Continuation) Your answers to the last problem can be turned into a new type of addition called addition modulo 12 or addition mod 12 for short. We can re-write the problems above as $6+_{12} 3=9,6+{ }_{12} 6=0$, and $6+_{12} 11=5$. Find the value of each of the following.
A. $3+{ }_{12} 4$
B. $9+{ }_{12} 9$
C. $11+{ }_{12} 10$
3. What are all the possible numbers that could be answers to an addition mod 12 problem? This set of numbers is called $\mathbb{Z}_{12}$ (read ' Z sub 12 ' or ' Z 12 ', for short). What numbers do you think you would find in a set called $\mathbb{Z}_{4}$ ? How about $\mathbb{Z}_{16}$, or $\mathbb{Z}_{29}$ ?
4. Draw a clock face that would help you solve problems in addition mod 7. Then explain how it can be used to find the standard names for each of the following.
A. $3+73$
B. $5+{ }_{7} 6$
C. $2+{ }_{7}(3+76)$
D. $\left(6+{ }_{7} 6\right)+_{7}\left(6+{ }_{7} 6\right)$
5. (Continuation) You're clock face can also be used to illustrate subtraction mod 7. Explain how. Create at least three subtraction mod 7 problems to illustrate your explanation.

## Operational Systems, Logic, \& Number Theory

1. To find the average distance between points in the figure below, you could measure the distance between every pair of points. Given that there are 190 possible distances, however, this may not be reasonable. Instead, select 10 pairs of points and use the average distance between them as an estimate. Work in millimeters. How do you think your average will compare to your classmates'?

2. (Continuation) The standard name of a number is that number written as a single numeral. Replace each triangle with standard names from $\mathbb{Z}_{12}$ that make each of the following equations true.
A. $\Delta+{ }_{12} 4=0$
B. $\Delta+{ }_{12} \Delta=0$
C. $10+{ }_{12}\left(\Delta+{ }_{12} 2\right)=5$
3. Show how you can compute each of the following using only regular addition and subtraction.
A. $3+{ }_{12} 8$
B. $5+{ }_{12} 9$
C. $6+{ }_{12} 1$
D. $10+{ }_{12} 11$
E. $x+{ }_{12} y$
4. Show how you can compute each of the following using only regular addition and subtraction.
A. $5-73$
B. $3-{ }_{7} 5$
C. $4-71$
D. $1-74$
E. $x-{ }_{7} y$

## Operational Systems, Logic, \& Number Theory

1. In order to promote the use of seat belts, car makers have made it so that a loud, annoying beeping sound goes off if the driver of the car isn't wearing their seatbelt. They do this by putting a sensor on the seatbelt that has a value of 'true' if the seatbelt is buckled but has a value of 'false' if the seatbelt is not buckled. This sensor is connected to a beeper. If the beeper has a value of 'false', it does nothing. If it has a value of 'true', it beeps. Which value for the seatbelt's sensor should cause the beeper to become 'true'? Which should cause it to become 'false'?
2. (Continuation) The relationship between the seatbelt's sensor and the beeper is an example of the logic operation called the negation. The negation, represented by the symbol $\sim$, takes one input, either true or false, and gives one output, again either true or false. If $P$ represents the value of the seatbelt sensor, then ${ }^{\sim} P$ represents the value of the beeper. Given the connection between the negation and the design of the seatbelt system, fill in the truth table for the negation operation below.

| $P$ | ${ }^{\sim} P$ |
| :---: | :---: |
| $t$ |  |
| $f$ |  |

3. The notation $[72]_{29}$ means 'the remainder when 72 is divided by 29 '. Find the value of this expression. Then find three possible values for $x$ that will make each of the following equations true.
A. $[x]_{29}=1$
B. $[x]_{7}=5$
C. $[5+x]_{29}=12$
D. $[2 \cdot x]_{7}=3$
4. (Continuation) Show how the remainder operation introduced in the last problem can be used to find the value of $4+_{7} 5$. Can the remainder operation help solve all modular addition problems? What about modular multiplication?
5. A circular bus route has 20 stops at intervals of 5 minutes. The route begins at 'Stop 0 ' and moves on to 'Stop 1', 'Stop 2 ', $\ldots$ and so on. At which stop will the bus be after 7.25 hours?
6. A family leaves for vacation on a Tuesday. On what day of the week will they return if they are gone 25 days? How about if they are gone 32 days? 41 days? 96 days?
7. if it is 10 am in new york city, what time is it in san francisco?. How about Denver? Why is there a difference?
8. The dots below represents the registered voters in Gridland. A dot is black if the person is a member of the Republican Party and is white if the person is a member of the Democratic Party.
(a) How many registered voters are there in Gridland?
(b) Devise a quick method for fairly selecting 20 voters at random and use your sample to estimate the total number of voters there are from each party.
(c) A local Republican politician believes that if he runs a series of TV commercials promoting his image, he can get $20 \%$ of the Democrats in Gridland to switch sides and vote for him. Assuming he is right, will this be enough for him to win the next election?


#### Abstract

400000000000000000••100000000••000000•100000000000000000000 00001000 1.            00000000000000000000000100001000001000000000          00000000000000000010000001000010000000000010000         4000.0000     0000000000001000000000000000010000010000000.10000000.     $0,0 \cdot 0 \cdot 0 \cdot 0 \cdot 1000 \cdot 100 \cdot 000000 \cdot 000000 \cdot 00000000000000$   


## Operational Systems, Logic, \& Number Theory

1. If you perform addition mod 12 on two numbers in $\mathbb{Z}_{12}$, you get another number in $\mathbb{Z}_{12}$. Because of this, we say that addition $\bmod 12$ is an operation on $\mathbb{Z}_{12}$. Is regular addition an operation on $\mathbb{Z}_{12}$ ? Is regular subtraction an operation on the counting numbers? Is regular multiplication an operation on the integers? Explain.
2. In military time, midnight is 0 o'clock, 3:00 AM is 3 o'clock and 4:00 PM is 16 o'clock. If it is currently 13 o'clock military time, what time will it be in 4 hours? 10 hours? 49 hours? $x$ hours?
3. As we all know, the sun only appears to rotate around the earth each day. In reality, it is the earth that is rotating with respect to a sun whose position is more or less fixed in place. The earth is divided into 24 time zones. Figure 3 below shows these zones in a simplified way. Through what angle must the earth rotate for the sun to pass over exactly three of those time zones?


Figure 3: Simplified Time Zones
4. length of equator. How wide is each time zone is at its widest point?
5. Let us name the time zones Time Zone 0 , Time Zone 1 , Time Zone 2,..., Time Zone 22, and Time Zone 23. If it is 14 o'clock military time in Time Zone 3, then it is 13 o'clock in Time Zone 2 and 15 o'clock in Time Zone 4. What time is it in Time Zone 23? Time Zone 11? Time Zone 18 ?
6. (Continuation) If it is 3 o'clock military time in Time Zone 12, what time will it be in 76 hours in Time Zone 3? How many total degrees will the earth have rotated in that 76 hours?

1. $(4+6) \bmod 12$. Find the value of each of the following.
A. $8+{ }_{12} 9$
B. $6-{ }_{12} 11$
C. $21 \times{ }_{29} 27$
D. $321 \times{ }_{29} 67$
2. The letters qei.elci were coded using the Key $\mathbf{5} \rightarrow$ Multiply by $\mathbf{2} \rightarrow$ Key $\mathbf{5}$ procedure. Decode them.
3. Is multiplication an operation on the even integers? Is multiplication an operation on the odd integers? Is mod 29 multiplication an operation on the even integers in $\mathbb{Z}_{29}$ ? For each, explain how you know and illustrate your answers with examples.
4. A subset of a population is called a sample of that population. The number of elements in the sample is called the sample size.
(a) What percentage of the squares in the grid below are black?
(b) Choose any 5 squares on the board. In other words, select a sample of squares whose sample size is 5 .
(c) Is the percentage of black squares in your sample close to the percentage of black squares in the whole grid? If so, your sample is said to be representative of the whole grid. If not, your sample is not representative.
(d) If you selected a representative sample in part (a), find a 5 square sample that is not representative. If you selected a sample that was not representative, find one that is.
(e) Are there any 5 -square samples that are perfectly representative of the whole grid? If so, find one. If not, explain why not.

5. The maximum operation takes two numbers and spits out which of the two is larger. Just like ' + ' is used for addition, ' $\uparrow$ ' is used for the maximum operation. For example, $6 \uparrow 9=9$ because 9 is larger than 6 . Find the output for each of the following. Each output should be a single number.
A. $15.292 \uparrow 15.929$
B. $\left(2+{ }_{7} 6\right) \uparrow\left(2+{ }_{12} 6\right)$
C. $\left(14 \cdot{ }_{29} 28\right) \uparrow(15 \cdot 2927)$
6. How many possible addition mod 3 problems are there using the numbers from $\mathbb{Z}_{3}$ ? List them all in a column on the left hand side of the page. What are all the possible answers to addition mod 3 problems? List them all in a column to the right of your first column. Now draw arrows from each problem to its answer. You have just created a mapping diagram for addition mod 3 on $\mathbb{Z}_{3}$. Draw a mapping diagram for subtraction mod 3 on $\mathbb{Z}_{3}$.

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1. A coffee maker has two sensors, sensor A and sensor B. Sensor A has a value of 'false' until it detects that the water in the machine is hot enough to brew the coffee. Once it is hot enough, the sensor switches to 'true'. Sensor B is connected to the start button on the machine. It has a value of 'false' until someone presses the start button, which changes the value to 'true'. Both of these sensors are connected to a brewing mechanism, which has a value of 'false' by default. If the value of the brewing mechanism ever changes to 'true', it begins the process of brewing the coffee.
(a) If the coffee maker is working properly, should a value of 'false' for sensor A and a value of 'true' for sensor B cause the value of the brewing mechanism to become 'true'?
(b) In general, which values for sensors A and B should cause the value of the brewing mechanism to become 'true'?
2. (Continuation) The relationship between the two sensors and the brewing mechanism in the coffee maker is an example of the logical operation called the conjunction. The conjunction, represented by the symbol ' $\wedge$ ', takes two inputs, both either true or false, and gives one output, again either true or false. Given the connection between the conjunction and the design of the coffee maker, fill in the truth table for the conjunction operation below.

| $A$ | $B$ | $A \wedge B$ |
| :---: | :---: | :---: |
| $t$ | $t$ |  |
| $t$ | $f$ |  |
| $f$ | $t$ |  |
| $f$ | $f$ |  |

3. What do you think we should do if both numbers in a maximum operation problem are the same ( $6 \uparrow 6$, for example)? Explain why your choice is a good one.
4. (population of new york city) / (area of new york city). Find the value of the expression below, which makes use of the minimum operation. What does the answer mean?

$$
[(\text { population of NYC }) \div(\text { area of NYC })] \downarrow[(\text { population of LA }) \div(\text { area of LA })]
$$

5. integers mod 7. Is $+_{7}$ an operation on $\mathbb{Z}_{7}$ ? How can you tell by looking at its addition table? Is ${ }_{7}$ an operation on $\mathbb{Z}_{7}$ ?
6. The factors of 60 are all the numbers that divide evenly into 60 . Find all the factors of 60. Then find all the factors of 48 . Which factors do they have in common? The largest of these is called the greatest common factor of 60 and 48 .

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1. A house is being wired for a simple security system. The house has two doors (door $A$ and door $B$ ). Each door has a sensor, which is set to the value 'false' by default, but switches to the value 'true' if a burglar tries to open the door. Each of these sensors is connected to a single alarm bell, whose value is also set to 'false' by default. If the alarm's value ever changes to 'true', then the bell rings out to warn the home owners and scare away the burglar. If the alarm system is working properly, which of the four possible true/false combinations for sensors A and B should cause the value of the alarm to become 'true'?
2. (Continuation) The relationship between the two sensors and the alarm is an example of the logical operation called the disjunction. The disjunction, represented by the symbol ' $V$ ', takes two inputs, both either true or false, and outputs one value, again either true or false. Given the connection between the disjunction and the security system design, fill in the truth table below.

| $A$ | $B$ | $A \vee B$ |
| :---: | :---: | :---: |
| $t$ | $t$ |  |
| $t$ | $f$ |  |
| $f$ | $t$ |  |
| $f$ | $f$ |  |

3. A toad and a bullfrog are sitting on nearby lilly pads, each making their croaking sounds at a different rate. The toad croaks every 5 seconds while the bullfrog croaks every 7 seconds. They just croaked at the exact same time. Assuming they maintain their steady croaking rates, how many more times will they croak at the exact same time in the the next ten minutes?
4. List the first ten multiples of 8 . List the first ten multiples of 6 . Which multiples do they have in common? The smallest of these is called the least common multiple of 6 and 8 .
5. Imagine a population of 100 , where 50 of the people are male and 50 are female.
(a) Explain why a perfectly representative sample exists for any even sample size.
(b) If a three person sample is chosen, how close can the percentage of males in the sample come to the actual percentage? What about a sample of 5 ? How about 7 ?
6. (Continuation) $6\lfloor\downarrow$ is symbolic notation for 'the least common multiple of 6 and 8 '. $60\rceil 48$ is symbolic notation for 'the greatest common factor of 60 and 48 '. Find values for x that will make each of the following sentences true.
A. $4\rfloor 15=x$
B. $100\rceil 80=x$
C. $3\rfloor x=27$
D. $x\rceil x=7$
7. Is $\left\rfloor\right.$ an operation on $\mathbb{Z}_{7}$ ? Is $\left\rceil\right.$ an operation on $\mathbb{Z}_{7}$ ? Explain using relevant examples.

## Operational Systems, Logic, \& Number Theory

1. Prime numbers are natural numbers that have exactly two factors. If you multiply two different prime numbers together, how many factors does the result have? Does it matter which two primes you choose?
2. Let P be the point $(5,4)$ and Q be the point $(5,8)$. Plot these points and then plot the point that is directly in between them. This new point is called the midpoint of P and Q.
3. Plot the points $(5,8),(9,3)$, and $(-6,-4)$. These points are called lattice points because both their $x$ and $y$ coordinates are integers. Now plot the points $(3.5,8),(3.2,3)$, and $(-6.2,-4.9)$. These points are not lattice points because at least one of their coordinates is not an integer. How many lattice points are their whose $x$ and $y$ coordinates are in $\mathbb{Z}_{6}$ ?
4. An ant has been placed at the point $(0,0)$ and is about to take a random journey. After every second, the ant will walk one unit in a particular direction: either straight up, straight down, straight to the left, or straight to the right. The direction he walks after each second will be up to you. To tell the ant which direction to move, generate a random integer from 1 to 4 . 1 means right, 2 means up, 3 means left, and 4 means down.
(a) Starting at $(0,0)$, make two moves with the ant. Where did your ant end up? What are all the possible places your ant could have ended up after two moves? Are all these places equally likely?
(b) Answer the same questions as part (a) for the ant's position after a three move random walk.
5. Write three sentences. To the best of your knowledge, one of them should be true, one should be false and the other should be neither. In general, what types of sentences can have a truth value and which cannot?
6. Write the opposite, or negation of each of the following sentences:
(a) The seatbelt is buckled.
(b) Claudia is not a dancer.
(c) Two plus thirteen is fifteen.
(d) Three to the second power is not nine.
7. An ant is standing on the point $(-4,7)$ and is only allowed to walk straight up, down, left or right. What is the minimum distance, in graph paper units, that the ant will have to walk to reach the point $(0,0)$ ? In general, if the ant is standing on $(x, y)$, what is the minimum distance to $(0,0)$, in terms of $x$ and $y$ ? By the way, this type of distance between two points in the coordinate plane is called a lattice distance.

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1. Let the variable $P$ represent the statement 'Forty-four is divisible by three' and the variable $Q$ represent the statement 'Thirteen is a prime number'. The statement 'Fortyfour is not divisible by three' can be represented symbolically as $\sim P$. The statement 'Forty-four is divisible by three and thirteen is a prime number' can be represented symbolically as $P \wedge Q$. The statement 'Forty-four is divisible by three or thirteen is a prime number (or both)' can be represented symbolically as $P \vee Q$. Write each of the following sentences in symbolic form.
(a) Thirteen is not a prime number.
(b) Thirteen is not a prime number and forty-four is divisible by three.
(c) Forty-four is divisible by three or thirteen is a prime number.
2. Let $P$ represent the statement 'Addition is an operation on the natural numbers' and $Q$ represent the statement 'Subtraction is an operation on the natural numbers'. Write English sentences that correspond to each of the following.
A. $P \wedge Q$
B. $\sim Q \vee Q$
C. $\sim P \wedge P$
3. Let $R$ represent the statement 'Five is an odd number' and $S$ represent the statement 'Six is an even number'. Which of the following do you think should be considered true?
A. $R \wedge S$
B. $R \wedge \sim S$
C. ${ }^{\sim} R \wedge S$
D. $\sim R \wedge \sim S$
4. did it rain in buffalo, ny on april 3, 2012? Jasmyn is planning an outdoor event on April 3, 2013 and wants to know if she needs to rent a tent to keep her guests out of the rain. Including 2012, how many years out of the last 10 did it rain on April 3 in Buffalo? In other words, based on the last 10 years, what is the probability that it rains on April 3 in Buffalo? Should Jasmyn rent the tent?
5. 2-dimensional random walk 1000 steps. Generate 20 different 2-dimensional, 1000 step random walks, each time recording the lattice distance between the starting point and ending point. Based on your sampling, what is the probability that this distance is less than 30 units?
6. Find the greatest common factor of 24,60 , and 90 . Find the least common multiple of 9,15 , and 21.
7. Since multiplication is an operation on the integers, we say that multiplication, $(\cdot)$, and the integers, $(\mathbb{Z})$, form an operational system. This is notated symbolically by putting the operator and the set in parenthesis with a comma in between them like this: $(\mathbb{Z}, \cdot)$. Think of five other operational systems and notate them symbolically.
8. 20 random numbers between 0 and 1. If you pick a random decimal between 0 and 1 and multiply that decimal by itself, what is the probability the result will be larger than 0.5 ? Estimate the answer by trying it 20 times. Is there another way to find the answer?

## Operational Systems, Logic, \& Number Theory

1. Since $1 \cdot x=x$ and $x \cdot 1=x$ for any integer $x, 1$ is said to be the neutral element of the operational system $(\mathbb{Z}, \cdot)$. It is important to note that it doesn't matter whether 1 is to the left or to the right of $x$. This is required to be considered a neutral element. Determine whether or not each of the following operational systems has a neutral element. If so, tell what it is. If not, explain why not.
A. $(\mathbb{N},+)$
B. $(\mathbb{Z},-)$
C. $(\mathbb{N}, \uparrow)$
D. $(\mathbb{N},\lceil \rceil)$
2. A self-avoiding random walk is similar to a regular random walk. The difference is that, in a self-avoiding walk, you are never allowed to visit the same point more than once. When simulating a self-avoiding random walk, if the random direction you generate will cause the walker to revisit a point, you simply generate a new direction. Starting at the point $(0,0)$ in the coordinate plane, simulate a self avoiding random walk by making 20 moves.
3. A certain air conditioning unit is programmed to turn on under two conditions: either the user turns it on or it detects that the room temperature is above $75^{\circ} \mathrm{F}$. The air conditioner also has an error light that turns on if there is something wrong with the temperature detection part of the system. Here's how the error light works: The air conditioner has two sensors, sensor A and sensor B. The value of sensor A is 'true' if the temperature is above $75^{\circ} \mathrm{F}$ but 'false' otherwise. The value of sensor B is 'true' if the air conditioner is on but 'false' if it is off. Both of these sensors are connected to the error light, which turns on if its value is 'true'. If the error light is working properly, which of the four possible true/false combinations for sensors A and B should cause the value of the light to become 'true'?
4. (Continuation) The relationship between the two sensors and the error light in the air conditioner is an example of the logical operation called the conditional. The conditional, represented by the symbol ' $\Rightarrow$ ', takes two inputs, both either true or false, and gives one output, again either true or false. Given the connection between the conditional and the design of the error light, fill in the truth table for the conditional operation below.

| $A$ | $B$ | $A \Rightarrow B$ |
| :---: | :---: | :---: |
| $t$ | $t$ |  |
| $t$ | $f$ |  |
| $f$ | $t$ |  |
| $f$ | $f$ |  |

5. Given that $a$ and $b$ are natural numbers, let $P$ be the phrase ' $a$ is even', $Q$ be the phrase ' $b$ is even', and $R$ be the phrase 'the product of $a$ and $b$ is even'. The formula $P \Rightarrow R$ is read 'if $a$ is even, then $b$ is even'. The conditional can be thought of as the 'cause and effect' connective because of its 'if this, then that' structure. Translate each of the following formulas into English.
A. $[P \wedge Q] \Rightarrow R$
B. $\left[{ }^{\sim} P \wedge Q\right] \Rightarrow R$
C. $[P \wedge \sim Q] \Rightarrow^{\sim} R$
D. $[P \wedge Q] \Rightarrow^{\sim} R$

Which of the above statements are true? Which are false?

## Operational Systems, Logic, \& Number Theory

6. Tyler wants to figure out what percentage of US states have a land area larger than 100,000 square miles but he doesn't want to check every state. Instead he takes out a map of the US and tosses 20 pieces of rice on it. Then he looks up the land area of each state that rice landed on. $40 \%$ of the states he looks up have a land area larger than 100,000 square miles. What do you think of the way Tyler selected the states he was going to look up? If you had to guess, would you think Tyler's percentage was too high, too low, or just about right? Explain.
7. Consider the equation $\left(a+_{2} b\right)+{ }_{2} c=a+{ }_{2}(b+2 c)$. If we are only allowed to substitute zeros and ones in for our three variables, one possible substitution would be $\left(0+_{2} 1\right)+_{2} 0=0+_{2}\left(1+_{2} 0\right)$. Another would be $\left(1+{ }_{2} 1\right)+_{2} 0=1+2\left(1++_{2} 0\right)$. Including these two, how many possible substitutions of zeros and ones are there? List them all.
8. (Continuation) Is $\left(0+_{2} 1\right)+_{2} 0=0+_{2}\left(1+_{2} 0\right)$ a true equation? How about $(1+21)+_{2} 0=1+{ }_{2}\left(1+_{2} 0\right)$ ? You should find that the rest of your combinations are true as well. Because the equation $(a+2 b)+_{2} c=a+2\left(b+{ }_{2} c\right)$ is true for all values in $\mathbb{Z}_{2}$, we say that the operational system $\left(\mathbb{Z}_{2},+_{2}\right)$ is associative. Is the operational system $(\mathbb{Z},-)$ associative? Provide evidence for your claim.
9. In football, a quarterback's completion percentage is the percentage of the quarterback's passes that are completed. It can also be viewed, however, as the probability that the quarterback completes a pass. Tom Brady's completion percentage is $65.6 \%$. To simulate a Tom Brady pass with the intention of determining whether or not he completes it, you could ask Wolfram|Alpha to generate a random integer between what two numbers? How could you simulate a series of five passes in a row?
10. Find at least one value for x that makes each of the following equations true.
A. $[x]_{7}+6=8$
B. $[x+6]_{7}=2$
C. $[3 \cdot x]_{7}=6$
D. $3 \cdot[x]_{7}=6$
11. It is currently 1 o'clock military time. What time will it be 167 hours from now? 384 hours from now? $x$ hours from now? Your formula for $x$ should work for 167 and 384 (or any other number of hours).
12. Because $0+x=x$ and $x+0=x$ for all $x \in \mathbb{Z}, 0$ is the neutral element of the operational system $(\mathbb{Z},+)$. Because the expression $3+\widehat{3}$ is equal to the neutral element of $(\mathbb{Z},+)$, 3 and $\widehat{3}$ are called inverses of each other within ( $\mathbb{Z},+$ ). Does every number have an inverse in $(\mathbb{Z},+)$ ?
13. Find a collection of prime numbers that multiply to give 15 . Now do the same for each of the following.
A. 21
B. 30
C. 50
D. 120

## Operational Systems, Logic, \& Number Theory

1. 2-dimensional self avoiding random walk 1000 steps. Because you are not allowed to revisit the same point twice in a self-avoiding random walk, it is almost guaranteed that the walker will eventually get stuck, having no possible place to move to. On average, approximately how long will it take for this to happen. Generate at least 20 examples to help you answer this.
2. The midpoint operation takes any two points in the coordinate plane as inputs and returns the midpoint of the two points. By plotting the points $P=(7,2)$ and $Q=(1,6)$, confirm visually that the point $M=(4,4)$ is the midpoint of $P$ and $Q$. Symbolically, we would write $P \rightarrow Q=M$ where ' $\theta$ ' is the symbol for the midpoint operation. Find each of the following. Be sure to plot all points to check that your answer is visually reasonable.
A. $(4,4)-\theta(10,4)$
B. $(0,0) \ominus(8,8)$
C. $(3,10)-\theta(9,0)$
D. $(2,3) \ominus(3,3)$
3. As we know, addition is an operation on the integers? Is the midpoint operation an operation on the lattice points whose $x$ and $y$ coordinates are in $\mathbb{Z}_{6}$ ? Provide an explanation for your answer.
4. Does 5 have an inverse? Explain why this question isn't specific enough to make any sense. Then make the question more specific and answer it. There is, of course, more than one way to make it more specific.
5. Consider the operational system $\left(\mathbb{Z}_{6},+_{6}\right)$. Find the inverse of each element of $\mathbb{Z}_{6}$ within this system. Because every element in the system has an inverse, the operational system $\left(\mathbb{Z}_{6},+_{6}\right)$ is said to be invertible. Is $\left(\mathbb{Z}_{6},{ }_{6}\right)$ invertible? Explain.
6. eli manning completion percentage. Approximate the likelihood that Eli Manning will complete his next 4 passes.
7. Is $(Z,-)$ invertible? To answer this question, Mirabel planned on checking a few different integers to see if they had inverses within $(Z,-)$. She quickly ran into a major problem, however. What problem did Mirabel run into? How do you think Mirabel should answer the question?
8. If $P \wedge Q$ is true, what can be said about $\sim[P \wedge Q]$ ? If $P \wedge Q$ is false, what can be said about $\sim[P \wedge Q]$ ? Use this information to construct a truth table for $\sim[P \wedge Q]$. Make sure your table shows the results of all four possible combinations of $P$ and $Q$.
9. random state. What percentage of states in the United States have a land area larger than 100,000 square miles? Approximate the answer using a sample size of 20.
10. What can be said about the truth value of each of the following?
A. $P \wedge^{\sim} P$
B. $\sim[P \wedge \sim P]$
C. $[P \wedge Q] \Rightarrow P$

## Operational Systems, Logic, \& Number Theory

1. Prove that $\left(\mathbb{Z}_{7},-_{7}\right)$ is not associative.
2. Design a fast method for randomly selecting a word from a book. Remember, to be random, each word in the book must have an equal chance of being selected.
3. A sample that is not selected randomly is said to be biased. A biased sample is selected in such a way that some members of the overall population are more likely to be selected than others. For each of the following biased sampling procedures, state what the overall population is. Then describe the subset of the overall population that is more likely to be selected than others.
(a) In order to determine what percentage of adults in his town watch Monday Night Football ${ }_{\circledR}$, Justin goes to the local sporting goods store and surveys the first 50 people who enter the store.
(b) In order to determine the average weight of a turkey, Hannah goes to a local farm and weighs 20 random turkeys on the farm.
(c) In order to determine the average height of a student in his middle school, Jamey firsts selects a grade level at random and then randomly selects 50 students from that grade level and measures their heights.
4. (a) Draw $\mathrm{a}+{ }_{4}$ table for $\mathbb{Z}_{4}$. (It should look like a multiplication table.)
(b) How can the neutral element of $\left(\mathbb{Z}_{4},+_{4}\right)$ be identified by looking at the table?
(c) Is $\left(\mathbb{Z}_{4},+_{4}\right)$ commutative? How can you tell by looking at the table?
5. Draw a picture of a clock with the numbers $0-11$ on it (Clock 12). How is each element of $\mathbb{Z}_{12}$ related to its inverse in $\left(\mathbb{Z}_{12},+_{12}\right)$ in terms of their positions on the Clock 12 face?
6. Consider the operator \&. This operator takes a left input and a right input and spits out the left input. So $3 \boldsymbol{\$} 8=3$ and $19 \% 4=19$.
(a) Is $(N, \boldsymbol{\%})$ commutative?
(b) Does $(N, \boldsymbol{\infty})$ have a neutral element?
(c) Is $(N, \boldsymbol{\infty})$ invertible?
7. Any natural number greater than 1 is either a prime number or can be expressed as the product of prime numbers. For instance, 60 is not prime but it can be expressed as $2 \cdot 2 \cdot 3 \cdot 5$. Identify the prime numbers in the following list. For those that are not prime, express them as the product of prime numbers. $\begin{array}{llllll}36 & 47 & 77 & 56 & 91 & 100\end{array}$

## Operational Systems, Logic, \& Number Theory

1. Using a sample size of 25 , approximate the probability that a 10 step 2-dimensional random walk ends up less than 6 lattice units from its starting point.
2. Consider the set P consisting of the numbers 2, 3, and 5. In roster form, we have $P=\{2,3,5\}$.
(a) How many subsets consisting of 0 elements does P have? List the subset(s).
(b) How many subsets consisting of 1 element does P have? List the subset(s).
(c) How many subsets consisting of 2 elements does P have? List the subset(s).
(d) How many subsets consisting of 3 elements does P have? List the subset(s).
(e) How many total subsets does P have?
3. (Continuation)
(a) Express 30 as the product of prime numbers.
(b) List all the factors of 30 . How many are there?
(c) Excluding the factor 1, express each factor of 30 as the product of primes (or just as the number itself if it is already prime.)
(d) Notice that your answer to the question in part (b) of this problem is the same as your answer to the question in part (e) of the last problem. This is not a coincidence. Use part (c) of this question to explain why.
4. Explain why the conjunction $(\wedge)$ is an operation on the set $\{t, f\}$, where $t$ and $f$ stand for true and false. Is this operational system commutative? Does it have a neutral element? Is it invertible?
5. The formula $P \Leftrightarrow Q$ represents the statement ' $P$ and $Q$ have the same truth value'. In general, if the connective $\Leftrightarrow$ could talk, it would say 'my left input and right input have the same truth value'. Below you will find three statements. One of them is definitely true and one of them is definitely false. As for the third, there is no way to tell without knowing the truth values of P and Q . Determine which is which.

$$
\sim^{\sim} P \Leftrightarrow P \quad \quad \sim P \Leftrightarrow Q \quad[P \wedge Q] \Leftrightarrow[Q \wedge P]
$$

6. There is a group of people. Each person has a different amount of money. They want to redistribute their money so that each person gets the same amount. Describe a mathematical process they could use to figure out how much each person should get.
7. (a) What is the sum of the first 2 odd numbers?; the first 3 odd numbers?; first 4 odd numbers?; 5 odd numbers?
(b) What is the sum of the first $n$ odd numbers?

## Operational Systems, Logic, \& Number Theory

1. The connective represented by the symbol ' $\Leftrightarrow$ ' is called the biconditional. Remembering that $\Leftrightarrow$ means 'my left input and right input have the same truth value', complete the truth table for the biconditional below.

| $P$ | $Q$ | $P \Leftrightarrow Q$ |
| :---: | :---: | :---: |
| $t$ | $t$ |  |
| $t$ | $f$ |  |
| $f$ | $t$ |  |
| $f$ | $f$ |  |

2. (a) Is the operational system $\left(\mathbb{Z}_{5},+_{5}\right)$ invertible? If so, find the inverse of every element in the system. If not, explain why not.
(b) Is the operational system $\left(\mathbb{Z}_{5},{ }_{5}\right)$ invertible? If so, find the inverse of every element in the system. If not, explain why not.
3. (a) If $\sim P$ is true, and $P \vee Q$ is true, what can be said about $Q$ ? Explain.
(b) If $P \Rightarrow Q$ is true, and $Q$ is false, what can be said about $P$ ? Explain.
(c) If $P \Rightarrow Q$ is true, and $P$ is true, what can be said about $Q$ ? Explain.
(d) If $P \wedge Q$ is true, what can be said about $P$ ? What can be said about $\sim Q$ ? Explain.
4. (a) When sampling is done, why is it important that the sample be representative of the entire population?
(b) Why does it make sense that biased samples tend not to be representative?
(c) Why does it make sense that random samples do tend to be representative?
(d) Does sample size affect how representative a random sample will tend to be?
5. (a) What can you say about the truth value of $[P \wedge Q] \Rightarrow P$ ? Explain.
(b) $(p$ and $q)$ implies $p$. Look at the truth table of $[P \wedge Q] \Rightarrow P$ and explain why it could have been predicted given your answer to part (a).
6. (a) How many factors does 70 have?
(b) How many subsets does the set $\{2,5,7\}$ have?
(c) Your answers to parts (a) and (b) should be the same. Explain why.
7. There are 4 ways to substitute t's and f's into the formula $P \Leftrightarrow Q$. The 4 ways are as follows: $t \Leftrightarrow t, t \Leftrightarrow f, f \Leftrightarrow t$, and $f \Leftrightarrow f$. How many ways are there to substitute t's and f's into the formula $P \Rightarrow[Q \vee R]$ ? List them all.
8. The prime factorization of 30 is $2 \cdot 3 \cdot 5$. Show that 30 has eight divisors (factors). Then find two other numbers that have eight divisors.
9. The Kelman's plan on having 5 children. Mr. Kelman wants at least 4 of them to be boys. Simulate the birth of 5 children, generating their gender at random. Did Mr. Kelman get his wish in your simulation? By repeating your simulation at least 15 times, determine the approximate probability that Mr. Kelman will get his wish.
10. Find six numbers that have four divisors. Write out the prime factorizations of each of your six numbers. Try to put your numbers into two different categories based on the structure of their prime factorizations.
11. The table below shows the beginning of a truth table for the formula $[P \wedge Q] \Rightarrow[P \vee R]$. Explain why it is necessary to include eight rows for this truth table. Finally, complete the truth table and explain why you could have predicted that its last column would contain all trues.

| $P$ | $Q$ | $R$ | $[P \wedge Q]$ | $[P \vee R]$ | $[P \wedge Q] \Rightarrow[P \vee R]$ |
| :---: | :---: | :---: | :--- | :--- | :--- |
| $t$ | $t$ | $t$ |  |  |  |
| $t$ | $t$ | $f$ |  |  |  |
| $t$ | $f$ | $t$ |  |  |  |
| $t$ | $f$ | $f$ |  |  |  |
| $f$ | $t$ | $t$ |  |  |  |
| $f$ | $t$ | $f$ |  |  |  |
| $f$ | $f$ | $t$ |  |  |  |
| $f$ | $f$ | $f$ |  |  |  |

4. Is $\left(\mathbb{Z}_{8},+_{8}\right)$ invertible? Prove it. Is $\left(\mathbb{Z}_{6},+_{8}\right)$ invertible? Prove it.
5. A tautology is a logical formula that is always true, no matter what truth values you put in for the variables. By looking at the truth tables for each of the following formulas, determine which of them are tautologies.
(a) ( $p$ and $q$ ) implies ( $p$ or $q$ ). $[P \wedge Q] \Rightarrow[P \vee Q]$
(b) $p$ equivalent $\sim(\sim p) . P \Leftrightarrow \sim[\sim P]$
(c) $[P \vee Q] \Rightarrow P$
6. Imagine you are given a list of numbers. To find the list's mean (sometimes called the list's average), you add the numbers together and divide by how many numbers there are.
(a) If two numbers in a list are 25 and 3 , find two other numbers that will make the mean of the list 20.
(b) Again starting with 25 and 3, find two other numbers that will make the mean of the list 8 .

## Operational Systems, Logic, \& Number Theory

1. Two out of the three statements below have the same meaning. Which two?
(a) ' $[P \wedge Q]$ and $[Q \wedge P]$ always have the same truth value.'
(b) ' $[P \wedge Q]$ and $[Q \wedge P]$ are both tautologies.'
(c) 'The formula $[P \wedge Q] \Leftrightarrow[Q \wedge P]$ is a tautology.'
2. Invent an operational system whose elements are $\{0,1,2,3\}$. Your operational system should have a neutral element but not be commutative. Make a table (like a multiplication table) for your operation.
3. The factors of 10 are $1,2,5$, and 10 . The sum of the factors of 10 , however, is 18 . Find the sum of the factors of $2,2^{2}, 2^{3}$, and $2^{4}$. Find a formula for the sum of the factors of $2^{n}$.
4. If you know that $P$ is true and that $P \Rightarrow Q$ is true, explain how you know for sure that $\sim[\sim Q]$ must be true.
5. Two numbers are relatively prime if their only common factor is 1 . Determine which of the pairs of numbers are relatively prime.
A. 12 and 63
B. 21 and 100
C. 128 and 1,764
D. 13 and 9,074
6. How many counting numbers less than 8 are relatively prime to 8 ?
7. Explain why $2+3=1$. Use the same reasoning to explain why $0+{ }_{3} 4=1$. What is the value of $13+{ }_{3} 22$ ?
8. If we know that $P \Rightarrow Q$ is true and that $P$ is true, what can be said about $Q$ ? This result is called Modus Ponendo Ponens, which means 'the way that affirms by affirming'. In other words, if you know $P \Rightarrow Q$ is true, and you can affirm that $P$ is true, then you can affirm that $Q$ is true. Modus Ponendo Ponens is almost always called by it's shorter name, Modus Ponens. It is the most basic conclusion that can be made in logic.
For each of the following, determine whether or not Modus Ponens is being used.
(a) If I know $A \Rightarrow B$ is true and I know $A$ is true, then I can conclude that $B$ is true.
(b) If I know $R \Rightarrow S$ is true and I know $S$ is true, then I can conclude that $R$ is true.
(c) If I know $[P \vee Q] \Rightarrow[R \vee S]$ is true and I know $[P \vee Q]$ is true, then I can conclude that $[R \vee S]$ is true.
9. It is true that if you flip a coin 10 times, the most likely outcome is 5 heads and 5 tails. How likely is this, however? Use simulation to answer this.
10. Why is it obvious that all of the following formulas have the same truth value?
$P \wedge Q$
$\sim[\sim P] \wedge Q$
$P \wedge \sim[\sim Q]$
$\sim[\sim P] \wedge \sim[\sim Q]$
11. How reliable are random samples of size 50 from a population of 500 ? To answer this question, Lauryn imagined a population of 500 people, half male and half female. To the males, she assigned the numbers 1 through 250 . To the females, she assigned numbers 251 through 500. Then she used Wolfram|Alpha to generate 50 random integers from 1 to 500 . The first time she did this, $44 \%$ of the integers were male. The second time, $52 \%$ of the integers were male. In all, she generated 100 sets of 50 random integers, each time recording the percentage of males in the sample. The results are shown in the dot plot below. Add five more dots to the plot after asking Wolfram|Alpha for the necessary samples. How do you think Lauryn should answer the question posed at the beginning of this problem?

## Percentage of Males within 50 Person Samples of a Population of 500


2. (Continuation) The 7th grade presidential election is coming up. Eddie, one of the two candidates, believes that right around $50 \%$ of the 7 th grade students are planning on voting for him. To check this, he selects 507 th graders at random and gives them a survey asking who they plan to vote for. Of these 50, 19 say they plan to vote for him. Is it possible that Eddie was right about his $50 \%$ prediction? If so, how likely is it? If not, explain why not.
3. How many counting numbers less than 10 are relatively prime to 10 ? Name the numbers.
4. List the factors of each of the following numbers: $1,4,9,16,25$, and 36 . Confirm that each of these numbers has an odd number of factors. What is it about these numbers that causes them to have an odd number of factors?
5. The numbers 4,9 , and 25 have exactly three factors. Find three other numbers that have exactly three factors. Write the prime factorizations of $4,9,25$ as well as the three numbers you found. What do all of these prime factorizations have in common?

## Operational Systems, Logic, \& Number Theory

1. The Hernandez family and the O'Reilly family both have 5 children. The Hernandez children's ages are $2,5,11,15$, and 22 . The O'Reilly children's ages are $8,10,11,12$, and 14. Verify that the mean child age is the same for both families. Despite this similarity, the ages for each family differ in a significant way? How do they differ?
2. Plot the numbers $2,5,11,12$, and 15 on a number line. Find the mean of these numbers and plot this as well. How far is each of the original numbers from the mean? What is the average distance between these numbers and the mean? Your answer to the last question is called the mean absolute deviation of the original 5 numbers. It is commonly used to measure how spread out a set of data is.
3. Make an operation table (just like a multiplication table) for each of the following. $\left(\mathbb{Z}_{2},+_{2}\right)\left(\mathbb{Z}_{3},+_{5}\right)\left(\mathbb{Z}_{5},+_{3}\right)$
4. (Continuation) Two of the above are operational systems and the other is not. Tell which is which and explain how you can tell by simply looking at your tables. Finally, determine whether or not the two operational systems are invertible.
5. Let's say you know that $R$ is true, and that $R \Rightarrow S$ is true, and that $S \Rightarrow^{\sim} Q$ is true. Prove that ${ }^{\sim} Q$ must be true.
6. Two of the three formulas below are equivalent to each other. Generate truth tables for each and use the tables to determine which two are equivalent. Does the negation distribute over the conjunction?
A. $\sim[P \wedge Q]$
B. ${ }^{\sim} P \wedge \sim Q$
C. $\sim^{\sim} P \vee \sim Q$
7. In early 2012 , half of the homes sold in Miami, FL sold for at or above $\$ 182,000$. This also means that half of the homes sold for at or below $\$ 182,000$. $\$ 182,000$, therefore, is the middle number in a list of Miami home prices. Such a middle value is called the median.
(a) median home price in san francisco. What is the median home price in San Francisco?
(b) number of homes in san francisco. How many homes are at or above the median? How many are at or below the median?
8. Josh has $\$ 6$ and Jeremy has $\$ 9$. Josh gives Jeremy $\$ 4$. How much money does each person have after this transaction? Explain why it would be impossible to answer this question if we could only use the operational system $(\mathbb{W},+)$.

## Operational Systems, Logic, \& Number Theory

1. If we know that $P \Rightarrow Q$ is true and that ${ }^{\sim} Q$ is true, what can be said about ${ }^{\sim} P$ ? This result is called Modus Tolendo Tollens, which means 'the way that denies by denying'. In other words, if you know $P \Rightarrow Q$ is true, and you can deny that $Q$ is true, then you can deny that $P$ is true. Modus Tolendo Tollens is almost always called by it's shorter name, Modus Tollens.
For each of the following, determine whether or not Modus Tollens is being used.
(a) If I know $A \Rightarrow B$ is true and I know ${ }^{\sim} A$ is true, then I can conclude that ${ }^{\sim} B$ is true.
(b) If I know $R \Rightarrow S$ is true and I know ${ }^{\sim} S$ is true, then I can conclude that ${ }^{\sim} R$ is true.
(c) If I know $[P \vee Q] \Rightarrow[R \vee S]$ is true and $\sim[R \vee S]$ is true, then I can conclude that $\sim[P \vee Q]$ is true.
2. Let's say you know that ${ }^{\sim} S$ is true, and that $R \Rightarrow S$ is true, and that ${ }^{\sim} R \Rightarrow P$ is true. Prove that $P$ must be true.
3. Invent an operational system consisting of four elements. Your operational system should be commutative and associative as well as have a neutral element. It should not, however, be invertible. Make an operation table to show the output for each pair of inputs in your system. Explain how you know your system meets all the criteria outlined in this problem.
4. Consider the made-up operator $\diamond$ which doubles its right input and adds the result to the left input. In other words, $a \diamond b=a+(2 \cdot b)$. Find the value of $2 \diamond 5,11 \diamond 12$, and $50 \diamond 121$.
(a) Is $(\mathbb{W}, \diamond)$ an operational system? Explain.
(b) Is $(\mathbb{W}, \diamond)$ commutative? Explain.
(c) Does $(\mathbb{W}, \diamond)$ have a neutral element? Explain.
5. Find all values of $x$ in $\mathbb{N}$ that make each of the following equations true.
A. $(3 \backslash x) \uparrow 7=24$
B. $[(x \downarrow 21) \uparrow 11]\lfloor 2=12$
6. Imagine a population of 200 , where $20 \%$ of the people prefer $\mathrm{Pepsi}_{\circledast \times}$ and $80 \%$ prefer Coke $_{\circledR}$. How reliable would a 60 person random sample be at predicting these percentages? In other words, how likely is it that a random 60 person sample would be representative of the entire population? Assign a number from 1 to 200 to each person and perform the sampling procedure 10 times to help you answer this question.
7. If you know that $[P \vee \sim[\sim P]] \Rightarrow Q$ is true, prove that $Q$ must be true. Use the words Modus Ponens in your explanation.
8. If you know that $P, Q$, and $R$ are all true, and you know that the formulas $[P \wedge Q] \Rightarrow S$ and $[S \wedge R] \Rightarrow T$ are also true, prove that $T$ must be true. Again, use Modus Ponens in your explanation.

## Operational Systems, Logic, \& Number Theory

1. The basic conditional forumla $P \Rightarrow Q$ has three important formulas that are often related to it. ${ }^{\sim} P \Rightarrow{ }^{\sim} Q$ is called the inverse of $P \Rightarrow Q$. The formula $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$. Finally, $\sim^{\sim} Q \Rightarrow^{\sim} P$ is called the contrapositive of $P \Rightarrow Q$. How is $P \Rightarrow Q$ changed to produce it's inverse, converse, and contrapositive? Answer this question using complete sentences.
2. How many numbers less than 3 are relatively prime to 3 ? Answer the same question for $5,7,11$, and 13 . If $p$ is a prime number, how many numbers less than $p$ are relatively prime to $p$ ? Explain.
3. Let $D_{12}$ be the set of divisors of $12, D_{8}$ be the set of divisors of 8 , and $D_{d}$ be the set of divisors of the greatest common factor of 8 and 12 .
(a) What is the value of $d$ in $D_{d}$ ?
(b) Write roster names for $D_{12}, D_{8}$, and $D_{d}$.
(c) Use set language to describe how $D_{12}, D_{8}$, and $D_{d}$ are related to each other.
4. If you list the integers from 1 to 657 , which number will be in the middle? How many numbers will be above the middle number? How many below? Because of your answers to the last two questions, what should this middle number be called?
5. (Continuation) If there are an odd number of numbers in an ordered list, there will always be a single middle number. Explain why.
6. If you know that $P$ is true and you know that ${ }^{\sim} Q$ is true, prove that ${ }^{\sim}[P \Rightarrow Q]$ must be true.
7. If you know that $[P \wedge Q] \Rightarrow[R \vee S]$ is false, what can be said about the truth values of each of the following formulas.

$$
[P \wedge Q] \quad[R \vee S] \quad P \quad Q
$$

8. An ant is walking on the number line. He starts at zero and walks in the positive direction. What is the first number the ant will reach that is not relatively prime to $13 ?$ What is the second number he will reach that is not relatively prime to 13 ? What is the third number he will reach that is not relatively prime to 13 ? What is the $n$th number he will reach that is not relatively prime to 13 ?
9. How many numbers less than 2 are relatively prime to 2 ? Answer the same question for $2^{2}, 2^{3}$, and $2^{4}$. How many numbers less than $2^{n}$ are relatively prime to $2^{n}$ ? Write a formula that will give you the answer no matter what counting number you substitute for $n$.
10. Remember that the Hernandez family children have ages 2, 5, 11, 15, and 22 and the O'Reilly children have ages $8,10,11,12$, and 14 . Before computing anything, predict which family will have the higher mean absolute deviation. Then check your prediction by computing the values.

## Operational Systems, Logic, \& Number Theory

1. Using the numbers in $\mathbb{Z}_{4}$, invent an operational system that is invertible but not commutative.
2. For each equation, find all numbers in $\mathbb{Z}_{6}$ that can be substituted for x to make the equation true.
A. $x{ }_{6} 3=3$
B. $x \cdot{ }_{7} x=x+{ }_{6} x$
3. Generate truth tables for $P \Rightarrow Q, Q \Rightarrow P,{ }^{\sim} P \Rightarrow{ }^{\sim} Q$, and $\sim^{\sim} Q \Rightarrow^{\sim} P$. Does a conditional have the same truth value as its inverse, converse, or contrapositive?
4. If you know that $P \Rightarrow Q$ is true and you know that $\left[{ }^{\sim} Q \Rightarrow^{\sim} P\right] \Rightarrow S$ is true, explain why $S$ must be true.
5. If $P \Rightarrow Q$ is true and $Q \Rightarrow S$ is true, must $P \Rightarrow S$ be true? Explain.
6. As a marketing strategy, Floaty Oats cereal decided to put a surprise toy action figure from the movie Toy Story $2_{\circledR}$ in each box. There are three action figures in all (Woody, Buzz, and Jessie), and there is an equal chance of getting each of the figures if you buy a box of cereal. Gary, a toy collector, wants to collect all three but only has enough money for four boxes of Floaty Oats. Use 20 simulations to approximate the probability that four boxes will be enough to get him his coveted full set of action figures.
7. List the counting numbers from 1 to 27 . Starting with the first number in your list (1), check each number in order to determine whether or not it is relatively prime to 27 . If it is relatively prime, leave it alone. If it is not, cross it out. You should see a pattern to the numbers you cross out. Explain how you could have predicted this pattern by examining the prime factorization of 27 .
8. Find all the numbers that are less than 14 and relatively prime to 14 . List them in order from least to greatest. Add the first number in your list to the last number in your list. Next, add the second number in your list to the second to last number in your list. Do the same for the third and the third to last number, etc. You should notice something. Will this always happen, or is the number 14 special? Try this with other numbers until you are convinced one way or the other.
9. The symbol $\vdash$ is called the turnstile. Mathematicians use it to communicate the fact that a certain logical formula must be true if others are true. For instance, we know that if $P$ is true and $Q$ is true, then $P \wedge Q$ is true. Using the turnstile, we can shorten this to: $P, Q \vdash P \wedge Q$. Prove that $P, Q, P \Leftrightarrow\left[{ }^{\sim} Q \vee R\right] \vdash R$.
10. How many positive multiples of 3 are less than or equal to 15 ? How many positive multiples of 3 are less than or equal to 33 ? How about 45? If $n$ is a multiple of three, how many positive multiples of 3 are less than or equal to it? Find a formula using $n$.

## Operational Systems, Logic, \& Number Theory

1. List the counting numbers from 1 to 25 . Starting with the first number in your list (1), check each number in order to determine whether or not it is relatively prime to 25 . If it is relatively prime, leave it alone. If it is not, cross it out. You should see a pattern to the numbers you cross out. Explain how you could have predicted this pattern by examining the prime factorization of 25 .
2. Imagine a population of 400 , where $65 \%$ of the people prefer Lays ${ }_{\text {© }}$ potato chips and $35 \%$ prefer Pringles ${ }_{\circledR \times 3}$. How reliable would a 15 person random sample be at predicting these percentages? In other words, how likely is it that a random 15 person sample would be representative of the entire population? Assign a number from 1 to 400 to each person and perform the sampling procedure 10 times to help you answer this question.
3. In $n$ is a multiple of 5 , how many positive multiples of 5 are less than $n$ ? Write a formula using $n$. Show that your formula works for at least two different multiples of 5 .
4. Write the prime factorization of six different positive 2-digit odd numbers. Then write the prime factorization of six different positive 2-digit even numbers. How can you tell whether a number is even or odd by looking only at it's prime factorization?
5. List the counting numbers from 1 to 16 . Starting with the first number in your list (1), check each number in order to determine whether or not it is relatively prime to 16 . If it is relatively prime, leave it alone. If it is not, cross it out. You should see a pattern to the numbers you cross out. Explain how you could have predicted this pattern by examining the prime factorization of 16 .
6. (Continuation) You have done the above excercise now for the numbers 16, 25, and 27. Use the knowledge gained from these three exercises to determine how many numbers less than 125 are relatively prime to 125 .
7. Modus Ponens can be represented symbolically as $P \Rightarrow Q, P \vdash Q$. Explain. Then use your knowledge of logic, the word 'conjunction', and the process of elimination to match each symbolic statement below with its corresponding name.
(a) $S, J \vdash S \wedge J$
(b) $S \wedge J \vdash S$
(c) $S \wedge J \vdash J$
(d) $S \Rightarrow J \vdash \sim^{\sim} J \Rightarrow^{\sim} S$
(e) $S \Rightarrow J, \sim J \vdash \sim S$
(f) $S \Rightarrow J, J \Rightarrow U \vdash S \Rightarrow U$
(g) $S \Leftrightarrow J, S \vdash J$

Modus Tollens
Syllogistic Inference
Contrapositive Inference
Right Conjunctive Simplification
Left Conjunctive Simplification
Conjunctive Inference
Modus Ponens for the Biconditional

## Operational Systems, Logic, \& Number Theory

1. If you list the integers from 1 to 224 , explain why there will not be a single middle number. You should be able to find two numbers in the middle, however. Calculate the average of these two numbers. This is the median. How many numbers are above the median? How many below?
2. The Venn Diagram below has four sections in it.
(a) Place each counting number lower than 24 in its appropriate section.
(b) Find all the numbers less than 24 that are relatively prime to 24 . Where are these numbers in the Venn diagram? Use the prime factorization of 24 to explain why.

Counting Numbers Less than 24

3. Prove each of the following:
A. ${ }^{\sim} Q, P \Rightarrow Q,{ }^{\sim} P \Rightarrow R \vdash R$
B. $P \wedge \sim Q, \sim R \Rightarrow Q \vdash P \wedge R$
4. The notation $\phi(24)$ means 'the number of counting numbers less than 24 that are relatively prime to $24^{\prime}$. Find the standard name of $\phi(24)$ as well as $\phi(18)$ and $\phi(27)$.
5. Show that $S \Rightarrow^{\sim} Q, Q,{ }^{\sim} S \Rightarrow[T \wedge U] \vdash[T \wedge Q] \wedge U$.
6. The home prices in Smalltown and Microville are listed below. Find the median home value for each town.

Smalltown: $\$ 224,000, \$ 82,000, \$ 224,000, \$ 135,000, \$ 112,000, \$ 130,000, \$ 112,000$
Microville: $\$ 500,000, \$ 120,000, \$ 121,000, \$ 112,000, \$ 120,000, \$ 225,000$
7. How many factors does $3 \cdot 7 \cdot 13$ have? What are they?
8. 'The contrapositive is the inverse of the converse'. Make sense of this statement.

## Operational Systems, Logic, \& Number Theory

1. The Venn Diagram below has four sections in it.
(a) Place each counting number lower than 20 in its appropriate section.
(b) Find all the numbers less than 20 that are relatively prime to 20 . Where are these numbers in the Venn diagram? Use the prime factorization of 20 to explain why.

Counting Numbers Less than 20

2. (Continuation) Draw a Venn Diagram like the one above to help you find $\phi(30)$.
3. How many factors does $2^{7}$ have? What are they?
4. The list below is an incomplete list of the numbers that are less than $n$ and relatively prime to $n$. Fill in all the blanks and find out what $n$ is.

$$
----,---, 5,---,---, 13,15,----,---, 23,------
$$

5. (a) Let $a$ stand for the number of positive multiples of 2 that are less than 40 ? Find $a$.
(b) Let $b$ stand for the number of positive multiples of 5 that are less than 40 ? Find $b$.
(c) Let $c$ stand for the number of positive multiples of 2.5 (also known as 10) that are less than 40 ? Find $c$.
(d) Calculate the following: $39-(a+b-c)$. By referring to the prime factorization of 40, explain why this calculation gives the value of $\phi(40)$.
6. (a) prime numbers near 100. Find the 20 prime numbers closest to 100 . Do the same for 1,000 and 100,000.
(b) mean deviation $\{53,59,61,67,71,73,79,83,89,97,101,103,107,109,113,127$, 131, 137, 139, 149\}. Find the mean absolute deviation for the 20 primes closest to 100 . Do the same for the other two sets of 20 primes. What do these results suggest about prime numbers as you move to the right on the number line?
7. Show that $P \Leftrightarrow[R \vee S],[R \vee S] \Rightarrow Q, P \vdash Q$.

## Operational Systems, Logic, \& Number Theory

1. How many factors does $3 \cdot 5 \cdot 11$ have? List them all. Then find 10 other numbers that have the same number of factors as $3 \cdot 5 \cdot 11$.
2. Below is an incomplete list of factors for a number, $n$. Find $n$. Is there more than one answer?
3. Prove that $S \wedge R, Q \Rightarrow T, R \Rightarrow^{\sim} T \vdash{ }^{\sim} Q \wedge S$. Use the phrases conjunctive simplification, modus ponens, modus tollens, and conjunctive inference in your proof.
4. For each row of dots below, add 3 vertical slashes so that the same number of whole dots fall between each consecutive pair of slashes. In the first row, which has been done for you, exactly one whole dot falls between each consecutive pair of slashes.

5. Show that $P \Rightarrow Q, R \Rightarrow S, Q \Rightarrow R \vdash P \Rightarrow S$. One particular logical tool from problem \#7 on page 27 is very useful in proving the above. Use this tool in your proof.
6. Draw a three circle Venn Diagram complete with a surrounding rectangle. Label your circles 'Multiples of 5 ', 'Multiples of 7 ', and 'Multiples of 11 '.
(a) Use this diagram to help you find $\phi(385)$.
(b) This Venn Diagram could have been used to help you find $\phi(n)$ for an infinite number of values of $n$. Name 3 of these.
7. Find each of the following:
A. $\phi(105)$
B. $\phi(221)$
C. $\phi(130)$
8. Given the history of rain in the past 10 years on April 28 in Buffalo, NY, what is the approximate probability that it will rain on at least 3 out of the next 4 April 28ths?
9. Using the elements of $\mathbb{Z}_{5}$, find or construct an operational system that is commutative and invertible. Make an operation table for your system and explain, in words, how your system meets the required crieria.

## Operational Systems, Logic, \& Number Theory

1. (a) Consider the list of numbers from 1 to 25 (which is $5^{2}$ ). What fraction of these numbers are not multiples of 5 ? Express your fraction in simplest form.
(b) Consider the list of numbers from 1 to 64 (which is $4^{4}$ ). What fraction of these numbers are not multiples of 4 ? Express your fraction in simplest form.
(c) Consider the list of numbers from 1 to 81 (which is $3^{4}$ ). What fraction of these numbers are not multiples of 3 ? Express your fraction in simplest form.
(d) Consider the list of numbers from 1 to $n^{m}$. What fraction of these numbers are not multiples of $n$ ?
2. Explain why the operator ' $\Rightarrow$ ' is an operation on the set $\{t, f\}$, where $t$ and $f$ stand for true and false. Is this operational system commutative? Does it have a neutral element? Is it invertible?
3. Below is a partially filled in two-column proof of $P, Q, R \wedge S,[P \wedge Q] \Rightarrow[R \Rightarrow T], \vdash T$. The left column contains two types of logic formulas: 1) Those that we are assuming to be true for the purposes of this proof (the ones to the left of the turnstile), and 2) those that we can deduce must be true based on the formulas we are assuming to be true. The right column contains the reasons that we are using to claim that the formulas in the left column are true. Fill in the missing information.
4. $P$
5. $Q$
6. $P \wedge Q$
7. $[P \wedge Q] \Rightarrow[R \Rightarrow T]$
8. $R \Rightarrow T$
9. 
10. Right Conjunctive Simplification (Formula 6)
11. T Modus Ponens (Formula 5, Formula 6)
12. If two sets of data have the same number of elements, the same mean, and the same mean absolute deviation, does that mean the sets of data have the exact same numbers in them?
13. List the numbers from 1 to $3 \cdot 5$ (or 1 to 15 if you don't know your multiplication facts).
(a) What fraction of these numbers are not multiples of 3? Express your fraction in simplest form.
(b) Now that we have gotten rid of the multiples of three, what fraction of the remaining numbers are not multiples of 5? Again, express your fraction in simplest form.
(c) Having gotten rid of the multiples of 3 and 5 , count the remaining numbers in your list. Explain why this count is the same as $\phi(15)$.
14. Use a two-column proof to prove the following: $P \wedge Q, P \Rightarrow S \vdash S$.

## Operational Systems, Logic, \& Number Theory

1. What is $\frac{4}{5}$ of 105 ? What is $\frac{6}{7}$ of 343 ? What is $\frac{10}{11}$ of 132 ?
2. The dot plots below show the ages of the players on the Pittsburgh Steelers and the Tampa Bay Buccaneers in August 2012.
(a) Just by glancing at the dot plots, which team appears to be younger in general?
(b) Compute the mean age for each team as well as the mean absolute deviation for the Steelers. Approximately how many mean deviations apart are the means for the two teams?

Ages of Pittsburgh Steelers Players in August 2012


Ages of Tampa Bay Buccaneers Players in August 2012


## Operational Systems, Logic, \& Number Theory

1. Perform each of the following calculations;
A. $\frac{2}{2}-\frac{1}{2}$
B. $1-\frac{1}{2}$
C. $\frac{5}{5}-\frac{1}{5}$
D. $1-\frac{1}{5}$
E. $10 \cdot\left(1-\frac{1}{2}\right) \cdot\left(1-\frac{1}{5}\right)$
2. If you know that $P \Leftrightarrow S$ is true, and you know that $Q \Leftrightarrow T$ is true, must $[P \wedge Q] \Leftrightarrow[S \wedge T]$ be true? Explain.
3. (a) A man named Augustus De Morgan claimed that $\sim[P \wedge Q]$ was equivalent to the formula ${ }^{\sim} P \vee \sim Q$. Find evidence to support this claim.
(b) De Morgan also claimed that ${ }^{\sim}[P \vee Q]$ was equivalent to ${ }^{\sim} P \wedge \sim Q$. What do you think?
4. List the numbers from 1 to $3 \cdot 7$ (from 1 to 21 , in other words).
(a) What fraction of these numbers are not multiples of 3? Express your fraction in simplest form.
(b) Now that we have gotten rid of the multiples of three, what fraction of the remaining numbers are not multiples of 7? Again, express your fraction in simplest form.
(c) Having gotten rid of the multiples of 3 and 7, count the remaining numbers in your list. Explain why this count is the same as $\phi(21)$.
5. (Continuation) Find the value of $21 \cdot \frac{2}{3} \cdot \frac{6}{7}$ and explain why this calculation gives you the value of $\phi(21)$. Then use a similar strategy to find $\phi(2 \cdot 7)$. Finally, does this strategy work for finding the value of $\phi\left(2^{2} \cdot 7\right)$ ? Explain.
6. The data below represent student scores on a final exam. As can be seen, the scores have been placed in order from least to greatest. Two vertical slashes have also been added at the front and back of the list. Add 3 more vertical slashes so that the same number of whole scores fall between each consecutive pair of slashes.

|  | 88 | 90 | 90 | 92 | 92 | 93 | 93 | 93 | 95 | 95 | 95 | 95 | 96 | 99 | 100 | 100 | 100 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |$|$

7. (Continuation) Use your three slashes above to generate three numbers using the following rules: 1) if the slash landed in between two test scores, the number for that slash will be the average of the two scores; 2) if the slash went through the middle of a score, the number for that slash will be the score it went through.
8. (Continuation) Of the three numbers you generated above, the left-most is called the $1 s t$ Quartile because one quarter of the data lie at or below this number.
(a) Using this same naming convention, what should the other two numbers be called? The middle number also goes by a different name that you should already know. What is it?
(b) The three numbers you have generated, combined with the minimum score (88) and the maximum score (100), make up what is known as the 5 Number Summary for the test scores. Find the 5 Number Summary for the test scores below. Don't forget to order the data first.
$\begin{array}{llllllllllllllll}79 & 88 & 74 & 74 & 91 & 72 & 79 & 84 & 84 & 74 & 79 & 91 & 91 & 100 & 96 & 91\end{array}$

## Operational Systems, Logic, \& Number Theory

1. Show that $R, R \Leftrightarrow[P \wedge Q],[P \wedge S] \Rightarrow^{\sim} P \vdash \sim P \vee^{\sim} S$ in two ways. First, make a two column proof using all the formal terminology (Modus Ponens, Conjunctive Simplification, etc). Second, explain the proof in your own words without using any formal terminology (you may use the names of connectives like conditional or disjunction).
2. phi (667). Try to find $\phi(23 \cdot 29)$ without using a computer. Be ready to explain your method. Then check your answer by putting the search string at the beginning of this question into Wolfram|Alpha.
3. For each of the following equations, find all whole number values for $q$ and $r$ that make the equation true:
A. $20=4 q+r$
B. $100=15 q+r$
C. $500=90 q+r$
4. Determine which of the following numbers are prime. If the number is not prime, write it's prime factorization.

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5. Explain how the tree diagram below proves that there are 8 possible outcomes when you flip a coin three times in a row. What do the H and T stand for? What are all the possible outcomes?

6. (Continuation) When you flip a coin three times, what are the chances of getting three heads? How about the chances of getting at least one tail?

1. (a) Find the sum of the counting numbers from 1 to 50 .
(b) Find the sum of the first 50 positive even numbers.
(c) Find the sum of the first 50 multiples of 5 .
2. One of the stories below illustrates modus ponens, one illustrates modus tollens, and one illustrates syllogistic inference. Tell which is which.
(a) If a restaurant adds more tables, they can seat more people. If they can seat more people, they can make more money. The moral of the story: more tables means more money.
(b) If you increase the pressure on a gas, the temperature of the gas will increase. A man held a small metal canister as a large amount of air was forced into it, increasing the air's pressure. His hand was burnt as a result.
(c) Rain is a form of precipitation. Wolfram|Alpha reports no form of precipitation on Monday, November 6, 2011 in Buffalo, NY. This means it didn't rain on November 6, 2011 in Buffalo, NY.
3. (Continuation) Make up a story like the ones above for contrapositive inference.
4. ( $\sim p$ or $q$ ) and $r$. What is the truth density of $(\sim P \vee Q) \wedge R$ ? What does this mean? How can you tell a formula's truth density by looking at it's truth table? What is the truth density of a tautology?
5. Generate 30 examples of 10 step 2-dimensional random walks. For each, write down the lattice distance between the starting point and the ending point. Do the same for 10 step 2-dimensional self-avoiding random walks.
(a) Construct two dot plots: one of the lattice distances for the regular 2-dimensional walk and one for the self avoiding walk. When compared, do your dot plots reveal a clear difference between the two walk types?
(b) For each type of walk, compute the mean as well as the mean absolute deviation of the lattice distances. Is there a difference in the means? If so, about how many mean deviations apart are they? Answer this last question twice: once for each of the mean deviations you computed.
(c) Which type of walk tends to produce an ending point that is further away from its starting point?
6. Write two-column proofs for each of the following:
(a) $P \Rightarrow Q, Q \Rightarrow R \vdash P \Rightarrow R$
(b) ${ }^{\sim} Q \Rightarrow \sim P, P \vdash Q$
(c) $R \Rightarrow S, \sim S \vdash{ }^{\sim} R \wedge \sim S$

## Operational Systems, Logic, \& Number Theory

1. Eddie is trying to find $\phi(36)$. He knows the prime factorization of 36 is $2^{2} \cdot 3^{2}$. He is having trouble deciding which of the following formulas will get him the right answer: $36\left(\frac{3}{4}\right)\left(\frac{8}{9}\right)$ or $36\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)$. Help him decide and be prepared to explain why your advice works.
2. In the expression $4 q+r$, the number 4 is called the coefficient of $q$. For each of the following equations, find all whole number values for $q$ and $r$ that make the equation true if $r$ must be less than the coefficient of $q$ :
(a) $30=5 q+r$
(b) $380=35 q+r$
(c) $421=23 q+r$
3. (a) prime factorization of 237,699 . Find the prime factorization of 237,699.
(b) Use the prime factorization you just found to write a formula for calculating $\phi(237,699)$.

Put your formula into Wolfram|Alpha and let it compute the answer for you.
(c) phi(237699). Ask Wolfram|Alpha for the phi of 237,699 directly. Do your answers to parts (b) and (c) agree?
4. If you rolled a 6 -sided die 120 times, how many 2 's would you expect to get?
5. (Continuation) The proportion $\frac{1}{6}=\frac{x}{120}$ is useful for answering the previous question. Explain what the numerator and denominator of each fraction mean in relation to the problem. Then solve the proportion and tell what its solution means.
6. Find the sum of the positive multiples of 3 less than 100 .
7. $\sim(p$ implies $q)$.
(a) Many formulas are equivalent to the formula $\sim[P \Rightarrow Q]$. One such formula uses only one $P$, one $Q$, and has the conjunction as it's principle connective. Find this formula.
(b) Prove the following: $\sim[P \Rightarrow Q], P \Rightarrow R \vdash R$.
8. Write a two-column proof for each of the following.
(a) $P \Rightarrow Q, P \vdash Q$
(b) $P \Rightarrow Q,{ }^{\sim} Q \vdash{ }^{\sim} P$
(c) $P \wedge Q \vdash P$
(d) $P \wedge Q \vdash Q$
(e) $P, Q \vdash P \wedge Q$
(f) $P \Rightarrow Q, Q \Rightarrow R \vdash P \Rightarrow R$
(g) $P \Leftrightarrow Q, P \vdash Q$
(h) $P \Leftrightarrow Q, Q \vdash P$

1. The information below represents the ages of the 15 players on the Los Angeles Lakers ${ }_{\circledR}$ and the New York Knicks $\circledR_{\circledR}$ as of May 1, 2011.

| Laker Ages(yrs.) | 31 | 31 | 31 | 25 | 32 | 23 | 22 | 21 | 36 | 30 | 26 | 31 | 38 | 35 | 31 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Knick Ages(yrs.) | 26 | 26 | 34 | 23 | 35 | 25 | 22 | 29 | 30 | 24 | 28 | 28 | 23 | 25 | 27 |

Find the five number summary for the Lakers and explain how it was used to make the box and whisker plot shown below.


Copy the box and whisker plot above onto graph paper. Then find the five number summary for the Knicks and use it to construct a second box and whisker plot. This second plot should be graphed on the same number line, above the Lakers' plot.
2. (Continuation) Reading Box and Whisker Plots
(a) Approximately what percentage of Lakers players are 31 or older?
(b) Approximately what percentage of Knicks players are 29 or younger?
(c) There are a lot of players in their early thirties on the Lakers. How does this affect their box and whisker plot?
(d) Which team would you say is 'younger'? Justify your answer using information from the plots. Is there another statistic you could calculate to help your argument?
3. Write a two-column proof of $S \wedge R, Q \Rightarrow T, R \Rightarrow^{\sim} T \vdash^{\sim} Q \wedge S$.
4. Consider the formula $P \Rightarrow[\sim Q \wedge R]$. If you randomly selected truth values for $P, Q$, and $R$, what is the probability the formula, as a whole, would be true? If you did this 4800 times, approximately how many times would the formula, as a whole, be true?

## Operational Systems, Logic, \& Number Theory

1. Let $P$ represent the statement 'logic formula $\mathcal{S}$ is equivalent to logic formula $\mathcal{J}$ ' and let $Q$ represent the statement 'logic formula $\mathcal{S}$ has the same truth density as logic formula $J^{\prime}$.
(a) What does the formula $P \Rightarrow Q$ represent in words? Is the statement true? Explain.
(b) Write the converse of $P \Rightarrow Q$ in words. Is the statement true? Explain.
(c) Write the contrapositive of $P \Rightarrow Q$ in words. Is the statement true? Explain.
2. Use Wolfram|Alpha to show that $P \Rightarrow Q$ is equivalent to ${ }^{\sim} P \vee Q$ in two different ways, as follows:
(a) First type each formula into Wolfram|Alpha seperately and obtain their truth tables. How do their truth tables show their equivalence?
(b) Generate the truth table for the formula $[P \Rightarrow Q] \Leftrightarrow\left[{ }^{\sim} P \vee Q\right]$. How does this truth table show that the formulas are equivalent?
3. (a) How many times does 35 go into 200 ? What is the remainder?
(b) Given that $r$ has to be smaller than the coefficient of $q$, find all whole number values for $q$ and $r$ that make the equation $200=35 q+r$ true.
(c) Given the connection between parts (a) and (b), what do you think the $q$ and $r$ stand for in the equation $200=35 q+r$ ?
4. Provide a two-column proof of $[Q \Rightarrow R] \wedge P,[Q \Rightarrow R] \Rightarrow^{\sim} T \vdash{ }^{\sim} T$.
5. random book. In general, are book titles longer, shorter, or about the same length as movie titles? Provide evidence.
6. (a) If $r$ has to be less than the coefficient of $q$, find all whole number values of $q$ and $r$ that make $183=28 q+r$ true.
(b) Mr. O'Malley's class has 28 numbered desks arranged in a circle. Each student is assigned a base number from 1 to 28 which indicates the desk number they are assigned to sit at on the first day of class. For every class after that, a random number is generated that each student must add (in mod 28) to their base number in order to determine where to sit for that day. If the random number for the day was 183 , where would the person whose base number is 1 sit?
(c) Explain the connection between parts (a) and (b).
7. Provide a two-column proof of $Q \Rightarrow R, \sim P \Rightarrow Q, \sim R \vdash P$.
8. (a) Find the product of $2^{3}$ and $2^{5}$ and express the answer as another power of 2 .
(b) Find the product of $6^{5}$ and $6^{2}$ and express the answer as another power of 6 .
(c) In general, if you multiply $b^{m}$ and $b^{n}$, what is the new answer expressed as a power of $b$ ?

## Operational Systems, Logic, \& Number Theory

1. (a) If you know that ${ }^{\sim} R \Rightarrow S$ is true, and you know that $S \Rightarrow[P \wedge Q]$ is true, and you know that $R \Rightarrow T$ is true, and you know that ${ }^{\sim} T$ is true, explain, in words, why $Q$ must be true.
(b) Provide a two-column proof of $\sim R \Rightarrow S, S \Rightarrow[P \wedge Q], R \Rightarrow T, \sim T \vdash Q$.
2. Mike and Sina, two fifth graders, have created a game of chance. In this game, there are two spinners. The first spinner is divided into quarters numbered $1,2,3$ and 4 . The second spinner is divided into sixths numbered $1,2,3,4,5$ and 6 . Draw a picture of these two spinners. How many unique outcomes are there in this game and what are they? The game is won if the sum of the numbers on the two spinners is greater than 7 . What are the chances of winning this game?
3. Mrs. Crubappel, a science teacher, has metal weights to aid in carrying out experiments. Some of the weights are 16 ounces, some are 8 ounces, some are 4 ounces, some are 2 ounces, and some are 1 ounce. What would a scale read if she put one 16 -ounce weight, four 8 -ounce weights, and 5 1-ounce weights on it? What is the minimum number of weights she would need to make the scale read:
A. 17 ounces
B. 25 ounces
C. 31 ounces
4. DeMorgan's Law. Provide a two-column proof of $\sim[P \vee Q], \sim P \Leftrightarrow R, S \Leftrightarrow^{\sim} Q \vdash S \wedge R$.
5. List the prime factorizations for the first six multiples of 18 . In general, how can you tell whether or not a number is a multiple of 18 just by glancing at its prime factorization?
6. Let's say we know that $S \wedge^{\sim} T$ is true, and that $S \Rightarrow^{\sim} R$ is true, and that ${ }^{\sim} P \Rightarrow Q$ is true.
(a) Patrick, a 9th grader, thinks that this makes $R \Rightarrow Q$ false. For the moment, assume he is correct. If he is correct, explain why $R$ must be true. And if $R$ is true, explain why $S$ must be false.
(b) Look carefully at what we knew to be true before Patrick came into the story and explain why $S$ can't be false.
(c) Explain why your contradictory finding from parts (a) and (b) prove that $S \wedge^{\sim} T$, $S \Rightarrow^{\sim} R, \sim P \Rightarrow Q \vdash R \Rightarrow Q$.
7. DeMorgan's Law. Provide a two-column proof of $\sim P, Q \Rightarrow P, R \Rightarrow P \vdash^{\sim}[R \vee Q]$.
8. Let $P$ represent the statement ' $x$ is an even number', $R$ represent the statement ' $x$ is a prime number', $S$ represent the statement ' $x$ is 2 '. Write a two column proof of the following. Then explain what the proof says about the statements above.

$$
[P \wedge R] \Rightarrow S, P, R \vdash S
$$

## Operational Systems, Logic, \& Number Theory

1. In Mr. O'Malley's subjective opinion, April 2011 seemed much colder than April 2010. The table below shows the average daily temperatures for both Aprils. Make a box and whisker plot for each one. Both should go above the same number line (be sure to label the number line). Use the plots to make an argument for or against Mr. O'Malley's vague memories.

| April 2010 | 61 | 66 | 67 | 53 | 56 | 54 | 58 | 52 | 37 | 41 | 50 | 46 | 50 | 50 | 56 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left({ }^{\circ} \mathrm{F}\right)$ | 51 | 40 | 42 | 46 | 49 | 50 | 44 | 47 | 56 | 50 | 52 | 44 | 46 | 49 | 63 |
| April 2011 | 34 | 35 | 36 | 51 | 34 | 33 | 40 | 45 | 49 | 57 | 62 | 47 | 43 | 44 | 40 |
| $\left({ }^{\circ} \mathrm{F}\right)$ | 48 | 38 | 34 | 36 | 43 | 38 | 41 | 54 | 49 | 50 | 59 | 67 | 57 | 43 | 51 |

2. Using your knowledge of logic terminology, explain what each of the following phrases must mean.
(a) contrapositive inference for the biconditional
(b) syllogistic inference for the biconditional
(c) modus tollens for the biconditional
3. (a) Find the quotient of $2^{5}$ and $2^{3}$ and express the answer as another power of 2.
(b) Find the quotient of $6^{7}$ and $6^{4}$ and express the answer as another power of 6 .
(c) In general, if you divide $b^{m}$ by $b^{n}$, what is the new answer expressed as a power of $b$ ?
4. (Continuation) Using the rule you found above, express the quotient of $8^{5}$ and $8^{5}$ as a new power of 8 . What is the standard name of $8^{5} \div 8^{5}$ ? Explain why it makes sense that $8^{0}=1$. Use a similar argument to explain why $b^{0}=1$ no matter what $b$ is.
5. The biconditional can be thought of as an operational system on the set $\{t, f\}$. Is this operational system commutative? Is it associative? Does it have a neutral element? Is it invertible?
6. If we know that $P \Leftrightarrow Q$ is true, and we know that $P \vee[Q \wedge R]$ is true, explain why $Q \vee[Q \wedge R]$ must be true and why $P \vee[P \wedge R]$ must be true.
7. Jocelyn has just spent the last hour acing the first 28 multiple-choice questions on her Algebra final. Surprisingly, though, she is absolutely clueless on the last two. Instead of leaving them blank, she decides to randomly choose one of the four choices for each question. What is the probability she gets them both right and escapes with a perfect paper?
8. (a) Write a two column proof of $\sim[R \vee S], S \Rightarrow[R \vee S] \vdash{ }^{\sim} S$. Your proof should be only three lines long, including your assumptions.
(b) Write a two column proof of ${ }^{\sim} P,[R \vee Q] \Leftrightarrow P, S \Rightarrow[R \vee Q] \vdash \sim[P \vee S]$.

## Operational Systems, Logic, \& Number Theory

1. Mrs. Crubappel has some metal weights to aid in carrying out experiments. Some of the weights are $10,000 \mathrm{mg}$, some are $1,000 \mathrm{mg}$, some are 100 mg , some are 10 mg , and some are 1 mg . What is the minimum number of weights she would need to make a scale read:
A. 3245 mg
B. $98,762 \mathrm{mg}$
C. $70,008 \mathrm{mg}$

You should have found a very quick way to determine the answers to the questions above. Why does this quick method work?
2. Find $\phi(1125)$ using only a scientific calculator. Show your work.
3. Show that 72 and 200 have the same number of divisors. Use their prime factorizations to explain why this is the case. Then find 5 other numbers that have 12 divisors.
4. You roll a pair of dice, one red, the other blue. Find each of the following probabilities:
(a) P (both dice show even numbers)
(b) P (both dice show a 5)
(c) P (red shows an odd number, blue shows a prime number)
(d) P (red shows a factor of 6 , blue shows a multiple of 2 )
5. Provide a two-column proof of $Q \Rightarrow S, R \Rightarrow^{\sim} S, P \Rightarrow R \vdash P \Rightarrow^{\sim} Q$.
6. (a) Explain why $[P \wedge Q] \Rightarrow[P \vee Q]$ is always true. In other words, explain why $[P \wedge Q] \Rightarrow[P \vee Q]$ is a tautology. Then name at least three other tautologies.
(b) Since tautologies are always true, they can be placed in the left column of any two-column proof. Make up a simple proof that makes use of a tautology.
7. Generate a truth table of the formula $[P \Rightarrow Q] \wedge[Q \Rightarrow P]$. This table should convince you that this formula is equivalent to a simpler, more common formula. What is this simpler formula?
8. (Continuation) Use the above result to help you write a two-column proof of $P \Leftrightarrow^{\sim} S$, $Q \Leftrightarrow S \vdash P \Rightarrow^{\sim} Q$.
9. Consider the statement $Q \Rightarrow^{\sim} P,^{\sim} R \Rightarrow[P \wedge Q] \vdash R$. Prove this statement by first assuming $R$ to be false, and then showing that this leads to impossible conditions. In other words, provide an indirect proof that $R$ is true by showing that it can't be false.
10. Provide a two-column proof of $Q \Rightarrow\left[{ }^{\sim} R \wedge S\right],\left[{ }^{\sim} S \vee^{\sim} T\right] \Leftrightarrow R, Q \vdash T$.

## Operational Systems, Logic, \& Number Theory

1. As you know, Mrs. Crubapple has weights that correspond to the powers of $2(1 \mathrm{oz}, 2 \mathrm{oz}$, $4 \mathrm{oz}, 8 \mathrm{oz}, 16 \mathrm{oz}, 32 \mathrm{oz}, 64 \mathrm{oz}, \mathrm{etc})$. What is the minimum number of weights required to make a scale read:
A. 23 ounces
B. 31 ounces
C. 170 ounces
D. 256 ounces
2. (Continuation) convert 23 to base 2. Ask Wolfram|Alpha to convert each of the four weights in the previous problem to base 2. Using your answers from the previous problem as a guide, explain how Wolfram|Alpha does the conversion. Prove you understand by converting 21 to base 2 without using Wolfram|Alpha.
3. factors of 15,379 . How many factors does 15,379 have? Use its prime factorization to systematically find at least five other numbers that have the same number of factors.
4. How can you tell if a number is a multiple of 84 just by glancing at its prime factorization? Give examples to support your conclusion.
5. Find the value of each of the folowing:
(a) $1 \cdot 2^{4}+1 \cdot 2^{3}+0 \cdot 2^{2}+1 \cdot 2^{1}+1 \cdot 2^{0}$
(b) $2 \cdot 8^{3}+5 \cdot 8^{2}+0 \cdot 8^{1}+5 \cdot 2^{0}$
(c) $5 \cdot 6^{4}+5 \cdot 6^{3}+5 \cdot 6^{2}+5 \cdot 6^{1}+5 \cdot 6^{0}$
6. Determine which of the following are true, with explanation.
(a) All prime numbers are odd.
(b) Some operational systems have a neutral element but are not commutative.
(c) All multiples of 24 have a 3 in their prime factorization.
(d) All the factors of 100 have a 2 in their prime factorization.
(e) There exists a natural number, $n$, such that $\phi(n)$ is odd.
7. (a) Consider the following collection of statements: 'If the basketball team's fast break runs well, the team will win. If the center rebounds well, the fast break will run well. So, if the team loses, the center did not rebound well'. Does the last sentence follow logically from the first two?
(b) Write a proof of the following: $F \Rightarrow W, R \Rightarrow F \vdash{ }^{\sim} W \Rightarrow^{\sim} R$.
(c) Explain how part (b) can be seen as a symbolic representation of part (a). In particular, explain what each of the variables in part (b) represents in part (a).
8. Does the disjunction distribute over the conjunction? In other words, is $P \vee[Q \wedge R]$ equivalent to $[P \vee Q] \wedge[P \vee R]$ ? Explain. Also, does the conjunction distribute over the disjunction? Explain.
9. Convert each of the following numbers to base 2 :
A. 86
B. 191
C. 395

## Operational Systems, Logic, \& Number Theory

1. The following collection of statements refers to a girl named Jenna who is playing a game in which she has to roll a pair of dice:
'If the sum on the dice is greater than $x$, then Jenna wins. If the sum on the dice is less than $y$, then Jenna does not win. When Jenna rolls the dice, the sum on the dice is less than $y$. Prove that the sum on the dice is not greater than $x$.'
(a) Let $P$ represent 'the sum on the dice is greater than $x$ '. Let $Q$ represent the statement 'the sum on the dice is less than $y$ '. Let $R$ represent the statement 'Jenna wins'. Write a symbolic proof statement, complete with logic formulas seperated by commas, as well as a turnstile, that represents the collection of statement above.
(b) Write a two column, indirect proof of your symbolic statement from part (a).
(c) Explain what each line of your two column proof means in terms of the story.
2. Each of the following numbers is written in base 2 (as indicated by the small subscript at the end of the number). Find the standard name of each. In other words, convert each of the numbers to base 10 .
A. $110101_{2}$
B. $100000000001_{2}$
C. $1111111_{2}$
3. (a) Provide evidence to support the fact that the formula $\sim[R \Rightarrow Q]$ is equivalent to the formula $R \wedge \sim Q$.
(b) Write an indirect proof of $S \wedge^{\sim} T, S \Rightarrow \sim R, \sim P \Rightarrow Q \vdash R \Rightarrow Q$.
4. Mrs. Crubapple has weights that correspond to the powers of 6 ( $1 \mathrm{oz}, 6 \mathrm{oz}, 36 \mathrm{oz}, 216$ oz, $1296 \mathrm{oz}, 7776 \mathrm{oz}, 46656 \mathrm{oz}$, etc). Using these weights, what is the minimum number of weights required to make a scale read:
A. 23 ounces
B. 452 ounces
C. 7775 ounces
D. 7776 ounces
5. (Continuation) convert 23 to base 6. Ask Wolfram|Alpha to convert each of the four weights in the previous problem to base 6. Using your answers from the previous problem as a guide, explain how Wolfram|Alpha does the conversion. Prove you understand by converting 425 to base 6 without using Wolfram|Alpha. Show your work.
6. We have seen how to convert standard numbers to base 2 as well as base 6 . Use this knowledge to explain how to convert standard numbers to base 3. Illustrate your procedure by showing how to convert 56 to base 3 .
7. I am thinking of a single digit in a number that is written in base 6 . What is the largest number that I could be thinking of? Explain why this is the case.
8. The disjunction can be thought of as an operational system on the set $\{t, f\}$. Is this operational system commutative? Does it have a neutral element? Is it invertible?
9. 1 moment. How many moments have we had together as a GMP class this year? (Which was your favorite?)

[^0]:    About technology: There are many problems in this text that require the use of Wolfram|Alpha in order to solve them. These problems can be identified in two ways: 1) the problem specifically mentions the use of Wolfram|Alpha; or 2) the problem begins with an italicized character string ending in a period. In the second case, the italicized character string is actual text input that can be put into the Wolfram|Alpha query bar in order to gain information needed for solving the problem. In addition to Wolfram|Alpha, the use of calculators as well as other computer software is not only encouraged, but may be mandatory based on the demands of the problem. Keep the following guidelines in mind when using technology: write before you calculate, so that you will have a clear record of what you have done; pay attention to the degree of accuracy requested; refer to your calculators online manual when needed; and be prepared to explain your method to your classmates.

